Technical Comments

Comment on "Dynamic Equations for Connected Rigid Bodies"

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A DYNAMICAL formalism for systems of rigid bodies, with the needs of digital simulation especially in mind, has recently been given by Sandler.[1] The philosophy he outlines is quite correct, though some elaboration of certain definitions might be desirable. I wish to comment that there already is a body of literature on this subject, including Refs. 2–6, and some of these carry the development further. (I do not cite the other formalisms of the same general type given in various unpublished works, i.e., company reports, during the past decade.) References 5 and 6, in particular, also were developed with a recognition of digital simulation problems.

I feel that Sandler's concluding remarks about "elastic bodies" are subject to misinterpretation, since this terminology usually is interpreted to refer to elastic continua to which his formulation (as well as the others cited here) does not apply.

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Reply by Author to R. E. Roberson

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I WOULD like to thank Professor Roberson for providing a list of very worthwhile references on the subject matter of my note.[1] I became aware of some of these in addition to Ref. 2 after submitting the note for final printing.

It was hoped that the discussion about elastic bodies would not lead to confusion since it was stated "...if it can be assumed that the body consists of lumped masses and massless springs." Modelling of elastic bodies by idealized deformable bodies such as massless springs has been very effective in structural dynamic analysis.

The analyses of Roberson's Refs. 5 and 6 are sufficiently general for application to certain classes of elastic bodies, namely those that can be modelled by rigid bodies connected by massless rotational springs. The discussion of elastic bodies in Ref. 1 was only intended to indicate that such analyses can be further generalized to include translational springs between rigid bodies. Unfortunately, the discussion appears to be inadequate.

A report[3] has been written which generalizes the analyses of Roberson's Refs. 5 and 6 to include idealized deformable bodies. The equations in the report are also applicable to bodies connected in a topological mesh configuration. The previous analyses have been restricted to topological tree configurations.

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Comment on "Jet Compression for Closed-Cycle Magnetoplasmadynamic Electrical Power Generation"

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Nomenclature

A = cross-sectional flow area, m^2
C = $\rho_3 u_3$, Eq. (7), kg$_f$-sec/m^3
c_p = specific heat at constant pressure, m^2/sec^2-°K
h = specific enthalpy, m^2/sec^2
M = Mach number
$\mathfrak{M}$ = molecular mass
$\dot{m}$ = mass flow rate, kg$_f$-sec/m
p = pressure or partial pressure, kg$_f$/m^2
$\mathcal{R}$ = universal gas constant, m^2/sec^2-°K
R = $\mathcal{R}/\mathfrak{M}$, specific gas constant, m^2/sec^2-°K
T = absolute temperature, °K
u = velocity, m/sec
x = quality
α = $\dot{m}_2/\dot{m}_1$, ratio of secondary to primary stream mass rate
Δh_v = heat of vaporization per unit mass, m^2/sec^2
ρ = density, kg$_f$-sec^2/m^4
$\varphi(T)$ = h_l, specific enthalpy of the liquid phase, m^2/sec^2
$\psi(T)$ = Δh_v, heat of vaporization per unit mass, m^2/sec^2
$X(T)$ = p_s, pressure of saturated vapor, kg$_f$/m^2

Subscripts

0 = related to stagnation conditions
1,2,3 = primary, secondary, and mixed stream, respectively

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l,g,v = liquid, gas, and vapor, respectively
r,s = reference and saturation, respectively

Introduction

To simplify a theoretical analysis, vapors often are assumed to behave like perfect gases. At moderate pressure this does not introduce significant errors as long as condensation does not occur. Referring to the problem of the jet compression of a stream of a permanent gas by means of a metal vapor driving stream, condensation is likely to take place in most cases of interest as it takes place practically always in nozzles of metal vapor turbines. In general, the amount of condensation is less than that occuring during an equilibrium expansion; then the perfect gas assumption is acceptable in the limiting case of a completely frozen process. However, if the working vapor is an alkali metal, the adiabatic expansion is in closer agreement with an equilibrium process than with a supersaturated-frozen process.[1,2] This comment aims to evaluate the departure of the results presented in Ref. 3, in which condensation was neglected from results obtained taking into account condensation.

Although not expressly stated, Eqs. (1) and (2) of Ref. 3 hold only if the driving vapor behaves like a perfect gas during the whole process of expansion both 1) prior to mixing and 2) at the end of mixing. As it will be shown, assumption 1, and in some cases also 2, is not met in practice.

Figures 1a and 1b report the entropy-enthalpy diagram of cesium vapor under the equilibrium and the perfect gas (frozen) assumption, respectively. A typical isentropic expansion from Ref. 3 exhibiting condensation of 15% of the vapor mass is shown.

The variations of temperature, specific volume, and enthalpy during this expansion are reported in Fig. 2. The departure of the two temperature curves even before the onset of condensation is due to the fact that in the equilibrium diagram of Fig. 1, a dimerization of Cs vapor molecules was taken into account, while in the diagram of Fig. 1b Cs was assumed to behave like a monatomic, perfect gas.

Notwithstanding the condensation of a substantial fraction of the original mass, the specific volume of the vapor-liquid mixture is considerably larger than that of the perfect gas due to the much lower temperature drop. Also the enthalpy drop (i.e. the kinetic energy at the beginning of mixing) is different in the two cases. Consequently the performance of the jet compressor is different.

Calculation of the Performance of the Jet Compressor

All of the assumptions made in Ref. 3 except the one according to which the vapor behaves like a perfect gas are retained, i.e., 1) the process is adiabatic; 2) c_{p_g} and c_{p_v} are constant; 3) frictional losses are negligible; 4) mixing is perfect at the end of the compression stage; and furthermore, 5) c_{pl} is constant; 6) when both liquid and vapor are present they are in equilibrium; 7) the noncondensed fraction of the vapor and the gas follow the perfect gas law; and 8) no slip exists between the liquid and the vapor or gas phase.

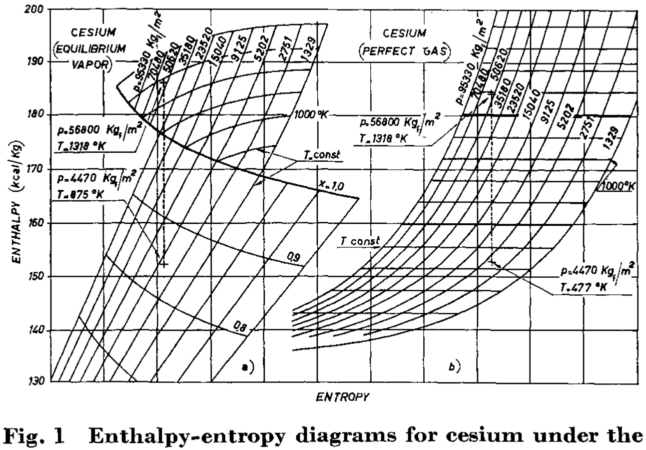

Fig. 1 Enthalpy-entropy diagrams for cesium under the equilibrium vapor and the perfect gas assumptions.

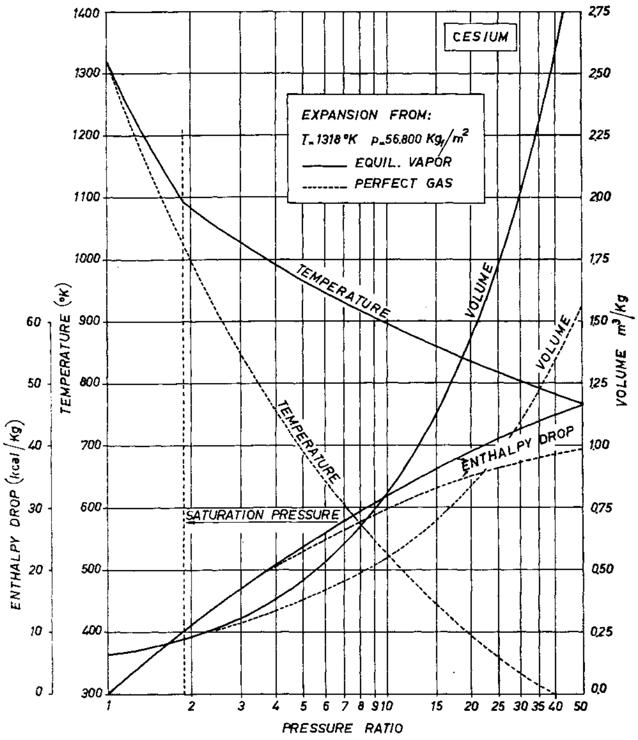

Fig. 2 Thermodynamic characteristics of equilibrium and frozen expansions.

Let subscripts 1 and 2 represent, respectively, the conditions of the driving vapor (Cs) and of the secondary stream (He) at the beginning of mixing; subscript 3, the conditions at the end of mixing. Under the aforesaid assumptions, the basic equations for conservation of mass, momentum, and energy in a constant area mixing process are

$$\dot{m}_1 + \dot{m}_2 = \dot{m}_3 \qquad (1)$$

$$\dot{m}_1 u_1 + p_1 A_1 + \dot{m}_2 u_2 + p_2 A_2 = \dot{m}_3 u_3 + p_3 A_3 \qquad (2)$$

$$\dot{m}_1(h_1 + u_1^2/2) + \dot{m}_2(h_2 + u_2^2/2) = \dot{m}_3(h_3 + u_3^2/2) \qquad (3)$$

besides

$$A_1 + A_2 = A_3 \qquad (4)$$

Through Eqs. (1) and (4), taking into account the definition of mass rate $\dot{m} = A\rho u$, Eqs. (2) and (3) yield

$$u_1 + p_1/\rho_1 u_1 + \alpha(u_2 + p_2/\rho_2 u_2) = (1 + \alpha)(C/\rho_3 + p_3/C) \qquad (5)$$

$$h_1 + u_1^2/2 + \alpha(h_2 + u_2^2/2) = (1 + \alpha)(h_3 + C^2/2\rho_3^2) \qquad (6)$$

in which ρ_1 is the density of the vapor-liquid mixture, $\alpha = \dot{m}_2/\dot{m}_1$ is the ratio of the secondary to the primary mass rate, and

$$C = \rho_3 u_3 = \rho_1\rho_2 u_1 u_2(1 + \alpha)/(\alpha\rho_1 u_1 + \rho_2 u_2) \qquad (7)$$

Quantities u_1, ρ_1, and h_1 are not easily related to the stagnation conditions of the driving stream because of the presence of the condensed phase. The unknown quantities p_3, ρ_3, and h_3 can be expressed as functions of p_3 and T_3. When this is done, the simultaneous solution of Eqs. (5) and (6) yields the stream conditions at the end of mixing.

Two different assumptions can be made: 1) during the mixing process the liquid phase vanishes and 2) at the end of mixing the liquid phase is still present. Which of the two stipulations is valid will be checked after the solution of the equations. In the first case taking into account the equation of state of a perfect gas $p = R\rho T$, and remembering that pressure p_3 is the sum of the partial pressures of the gas and

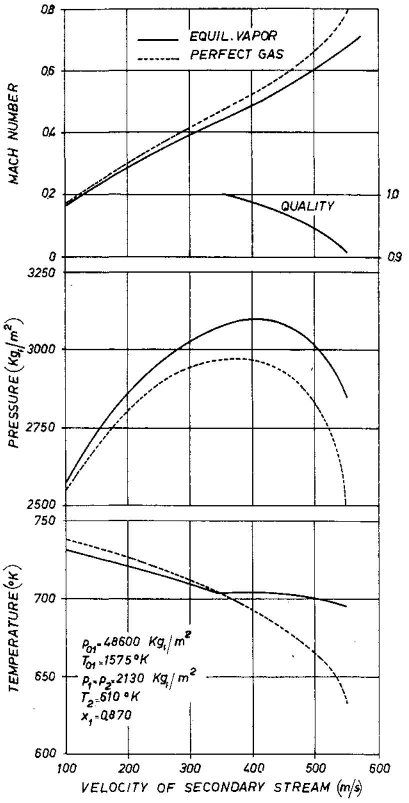

Fig. 3 Characteristics of the mixed stream under the equilibrium and under the frozen assumption.

of the vapor (p_{3g} and p_{3v}) which in turn are proportional to the number of moles present in the secondary stream, one has

$$\rho_3 = \rho_{3g} + \rho_{3v} = \frac{p_3}{T_3(\mathfrak{M}_2 + \alpha\mathfrak{M}_1)}\left(\frac{\alpha\mathfrak{M}_1}{R_2} + \frac{\mathfrak{M}_2}{R_1}\right) \tag{8}$$

$$h_3 = (\alpha h_{3g} + h_{3v})/(\alpha + 1) \tag{9}$$

in which $h_{3g} = c_{pg}T_3$, and $h_{3v} = h_{rv} + c_{pv}(T_3 - T_r)$. It can be observed that h_2 and h_1 in Eq. (6) must be based on the same reference enthalpy as h_{3g} and h_{3v} (zero and h_{rv}, respectively).

Through Eqs. (5–9) the static pressure and temperature at the end of mixing, p_3 and T_3, can be computed; the density ρ_3 and the velocity u_3 are then calculated through Eqs. (8) and (7), respectively. The problem of the mixing of two streams of perfect gases is solved by the same set of equations, provided that the quantities u_1, ρ_1, and h_1 are given values consistent with the perfect gas assumption. The starting stipulation predicting the complete vaporization of the liquid phase is acceptable if

$$p_{3v} = p_3\mathfrak{M}_2/(\mathfrak{M}_2 + \alpha\mathfrak{M}_1) \leq p_s \tag{10}$$

in which p_s is the equilibrium vapor pressure at temperature T_3, and $p_3 = p_{3g} + p_{3v}$. In case $p_{3v} > p_s$ some condensation will occur, and a new set of equations must be used to solve the mixing problem. Let x be the vapor quality defined as usual as the vapor mass fraction (the presence of the noncondensable gas does not affect this definition). According to the perfect gas law, partial pressures are proportional to the number of moles present in the gas state:

$$p_{3g}/p_{3v} = \dot{m}_2\mathfrak{M}_1/\mathfrak{M}_2\dot{m}_1x_3 = \alpha\mathfrak{M}_1/\mathfrak{M}_2x_3 \tag{11}$$

Thus,

$$x_3 = p_{3v}\alpha\mathfrak{M}_1/(p_3 - p_{3v})\mathfrak{M}_2 \tag{12}$$

If $\rho_{3v} = p_{3v}/R_1T_3$ is the vapor density not taking into account the condensed phase, and $\rho_{3g} = p_{3g}/R_2T_3$ is the gas density, then

$$\rho_3 = \rho_{3g} + \rho_{3v}/x_3 = (p_3 - p_{3v})(1 + \alpha)/R_2T_3\alpha \tag{13}$$

The specific enthalpy of the mixture, taking into account Eq. (12), can be written

$$h_3 = [\alpha h_{3g} + h_{3l} + p_{3v}\alpha\mathfrak{M}_1\Delta h_v/(p_3 - p_{3v})\mathfrak{M}_2]/(\alpha + 1) \tag{14}$$

in which $h_{3g} = c_{pg}T_3$, $h_{3l} = \varphi(T_3)$, and $\Delta h_v = \psi(T_3)$. Function φ is such that the reference value for enthalpy $h_{3v} = h_{3l} + \Delta h_v$ is the same as for enthalpy h_1 of Eq. (6). Since we admit that the liquid and the vapor phase are in equilibrium,

$$p_{3v} = p_{3s} = X(T_3) \tag{15}$$

The consistency of our starting assumption is checked through the relationship $x_3 \leq 1$. The simultaneous solution of Eqs. (5) and (6), taking into account Eqs. (7, and 13–15), gives the values of pressure and temperature at the end of mixing. The other characteristics of the flow are easily computed.

Results of the Numerical Analysis

The problem of the mixing of a Cs vapor and a He stream was studied by solving the foregoing equations numerically by means of a digital computer. The same starting values reported in Ref. 3 were adopted and are indicated for convenience in Figs. 3 and 4. Functions φ, ψ, and X were obtained from Ref. 4. Figure 3 gives T_3, p_3, and M_3 of the mixed stream as functions of u_2. The vapor quality at the beginning of mixing is 0.870. For velocities of the secondary stream lower than about 350 m/sec, the liquid phase completely vanishes during the process of mixing. The pressure rise is higher than predicted by the perfect gas assumption. Because of the vaporization of the liquid phase, the static temperature decreases from about 810°K to 700–750°K in the vapor assumption whereas it increases from 450°K to 650–750°K in the perfect gas stipulation. For secondary stream velocities >350 m/sec, the liquid phase is present even at the end of mixing. The reduction of the amount of liquid which vaporizes accounts for the leveling of T_3. In the example on Fig. 4, even for low u_2 the liquid

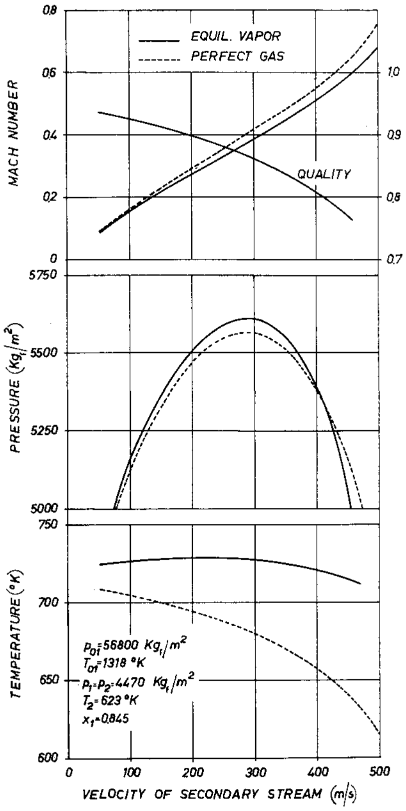

Fig. 4 Characteristics of the mixed stream under the equilibrium and under the frozen assumption.

phase does not vanish during the mixing process. In this case, too, the presence of the liquid fraction with its capacity to absorb or release large amounts of heat through the change of phase levels the T_3 curve. The differences in pressure rise between the vapor and the perfect gas are smaller than in the previous case.

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Reply by Author to G. Angelino

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I APPRECIATE Mr. Angelino's comment on the neglect of cesium vapor condensation effect which appeared to exist in some test runs in my paper, "Jet Compression for Closed-Cycle Magnetoplasmadynamic Electrical Power Generation." My paper was primarily a report of the results of an experimental investigation of the feasibility of using a stream of metallic vapor of high molecular weight to entrain and compress a gaseous stream of low molecular weight. The objective of the investigation was the experimental proof of the applicability and not a theoretical treatise of the jet compression process. Since the experimental set-up employed an expansion nozzle of fixed geometry and experiments were carried out at different stagnation temperatures and pressures of the primary driving stream, a theoretical treatment of the jet compression process has to take into consideration not only the change of state of fractions of the working medium, but also the effect of underexpansion and overexpansion of the primary stream. None of such effects can really be accounted for without extensive experimental investigations.

It is well known that a vapor stream, unless disturbed, can remain fully in the vapor state under certain degrees of cooling below its saturation point. In our experimental set-up both the helium gas and the cesium system were thoroughly purged and evacuated before experimentation so as to avoid condensation by nucleation. Experiments carried out at Massachusetts Institute of Technology Lab on condensation of CO_2 and ammonia have indicated wide discrepancies in degree of supersaturation for different vapor streams expanding through convergent-divergent nozzles.[1,2] It was also observed that the more abrupt the expansion, the higher the degree of supersaturation of the expansing stream. Presumably, the molecular weight of the medium will also affect the degree of supersaturation due to difference in diffusion velocity.

Our own experience with Cs has indicated that Cs vapor in mixture with He is very difficult to condense. This was evidenced in both the MPD electrical power generation[3] experiment and the jet compression experiment. In the former case, a straight tubular type of condenser was used. However, cesium deposits in large quantity were found passing through the cooler and condenser into the He surge tank and eventually the helium compressor. In the latter program, a multitube multibaffled-type mineral-oil-cooled condenser was used. Again a large amount of Cs was condensed in the He surge tank and the heater pipe. This strongly suggests the complexity of condensation of Cs vapor. The lack of information about condensation of Cs in supersonic nozzle flow and in mixtures of Cs with noble gases negates a realistic performance analysis without extensive experimental investigation.

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Comments on "An Improved Finite-Difference Method for Heat-Transfer Calculation"

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GROSS[1] developed an equation for the time step interval to be applied to the explicit finite-difference technique used in transient heat-transfer calculation. Briefly, his development is as follows:

$$t_i{}^+ = t_i + \frac{\Delta\tau_i}{C_i}\left[\sum_j K_{ji}(t_j - t_i) + S_i\right] \tag{1}$$

$$t_i{}^* \left(= \sum_j K_{ji}t_j + S_i\right)\Big/\sum_j K_{ji} \tag{2}$$

where $t_i{}^+$ = estimate of temperature of node i at the "new" time $\tau + \Delta\tau_i$; t_i = temperature of node i at the "old" time τ; C_i = thermal capacity of node i; K_{ji} = thermal conductance between node i and j nodes; S_i = energy source at node i; $t_i{}^*$ = temperature of node i if node i is in thermal equilibrium with neighboring nodes (j nodes); and $\Delta\tau_i$ = time step. The criterion for stability is given by Gross as:

$$t_i{}^+ \leq t_i{}^* \tag{3}$$

which he states results in the well known equation,

$$\Delta\tau_i \leq C_i\Big/\sum_j K_{ji} \tag{4}$$

If a small error in temperature is allowed, Gross writes

$$t_i{}^+ \leq t_i{}^* + \Delta t \tag{5}$$

which he states results in the following criterion:

$$\Delta\tau_i \leq \frac{C_i}{\sum_j K_{ji}}\left[1 + \frac{\Delta t \sum_j K_{ji}}{\sum_j K_{ji}(t_j - t_i) + S_i}\right] \tag{6}$$

Note there is an obvious typographical error in Gross's paper in that the

$$\sum_j K_{ji}$$

in the second expression in the bracketed term of Eq. (6) [Eq. (10) in Gross's paper] is missing.

The objection to the development of Gross's is that the time step stability criterion, Eq. (6), can result in a time step less than that given by Eq. (4). This is possible since the

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